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Some Statistical Parameters Related to the Nakagami-Rice Probability Distribution

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Formulas and tables are given for the mean and standard deviation of $R=20\log_{10} r$ where the random variable r has the Nakagami-Rice distribution. This distribution is of interest in connection with the short-term fading characteristics of some received radio fields. A particularly simple formula for the mean of R is obtained in terms of the well-known exponential integral function $-Ei(-x)$. Additional information concerning the median and interdecile range of R is also given.

1. Introduction

In certain radio propagation problems the field strength at a receiver may be approximated by the vector sum of a constant vector and a Rayleigh-distributed vector [Norton, Vogler, Mansfield, and Short, 1955]. If it is assumed that the Rayleigh-distributed vector has an rms amplitude k and that the constant vector has an rms amplitude of unity, then the probability distribution of the resultant r of their sum as given by Nakagami [1940] and Rice [1944, 1945] may be put in the form

$$P(r>r_0)=\frac{2}{k^2}\int_{r_0}^{\infty} r \exp\left[-(1+r^2)/k^2\right]I_0(2r/k^2)dr \tag{1.1}$$

where $I_0(x)$ is the modified Bessel function of first kind and order zero. If the Rayleigh distributed vector has an rms amplitude k_1, the constant vector has an rms amplitude k_2, and the resultant of their sum is r_1, then (1.1) gives the distribution of $r=r_1/k_2$ as a function of $k=k_1/k_2$.

The primary purpose of this paper is the determination of the mean $\overline{R}$ and standard deviation σ_R of $R=20\log_{10} r$. Additional information concerning the median $R(0.5)$ and interdecile range $R(0.1)$–$R(0.9)$ of R is taken from the paper by Norton et al. [1955]. The phase ϕ of the vector sum of a Rayleigh-distributed vector and a constant vector is discussed by Norton, Shultz, and Yarbrough [1952].

2. Calculation of $\overline{R}$ and σ_R

We will determine σ_R from the relation

$$\sigma_R^2=\overline{R^2}-(\overline{R})^2,$$

where $\overline{R^2}$ is the mean of R^2. Making use of (1.1), we find that

$$\overline{R}=\frac{2}{k^2}\int_0^{\infty}(20\log_{10} r)r \exp\left[-(1+r^2)/k^2\right]I_0(2r/k^2)dr \tag{2.1}$$

and

$$\overline{R^2}=\frac{2}{k^2}\int_0^{\infty}(20\log_{10} r)^2 r \exp\left[-(1+r^2)/k^2\right]I_0(2r/k^2)dr. \tag{2.2}$$

In appendix 1 of this paper, these two integrals are evaluated and the three equations which follow are derived.

$$\overline{R}=(10\log_{10} e)\left[-Ei(-1/k^2)\right]. \tag{2.3}$$

For $k \geq 1$, use

$$\sigma_R^2 = (10 \log_{10} e)^2 \left\{ \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1/k^2)^{n+1}}{(n+1)!(n+1)} \left(\sum_{j=1}^{n} \frac{1}{j} \right) - \left[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(1/k^2)^n}{n!n} \right]^2 \right\}. \qquad (2.4)$$

For $k < 1$, use

$$\sigma_R^2 = (10 \log_{10} e)^2 \left\{ -2Ei(-1/k^2)E^*(1/k^2) + 4 \int_1^{1/k^2} \frac{-Ei(-t)}{t} dt \right.$$

$$\left. -2 \int_1^{1/k^2} \frac{e^t[-Ei(-t)]}{t} dt - [Ei(-1/k^2)]^2 + 4Ei(-1/k^2)[\gamma + \ln (1/k^2)] + C_1 \right\}. \qquad (2.5)$$

In these equations,

$$-Ei(-x) = \int_x^{\infty} \frac{e^{-t}}{t} dt, \qquad x > 0;$$

$$E^*(x) = -\,⨍_{-x}^{\infty} \frac{e^{-t}}{t} dt, \qquad x > 0,$$

where

$$⨍_{-x}^{\infty} = \lim_{\epsilon \to 0} \left(\int_{-x}^{-\epsilon} + \int_{\epsilon}^{\infty} \right)$$

with $\epsilon > 0$; and $C_1 = 1.099019$, a constant. The functions $-Ei(-x)$ and $E^*(x)$ are well-known functions for which tables are available. See, for example, the book "Tables of Sine, Cosine, and Exponential Integrals," U.S. National Bureau of Standards [1940]. The integrals

$$\int_1^{1/k^2} \frac{-Ei(-t)}{t} dt$$

and

$$\int_1^{1/k^2} \frac{e^t}{t} [-Ei(-t)] dt$$

are evaluated in appendix 2 to six significant digits by numerical integration.

Letting $K = 20 \log_{10} k$, we can get the following asymptotic formulas from (2.3), (2.4), and (2.5).

$$\text{For } K > 20,\ \overline{R} \approx K + \frac{4.3429}{k^2} - 2.5068 \qquad (2.6)$$

TABLE 1

K	$R(0.5)$	$\overline{R}$	$\overline{R} - R(0.5)$	σ_R
−40	0.000	0.000	−0.000	0.061
−35	.001	.000	−.001	.109
−30	.002	.000	−.002	.194
−25	.007	.000	−.007	.346
−20	.022	.000	−.022	.616
−18	.034	.000	−.034	.776
−16	.054	.000	−.054	.980
−14	.086	.000	−.086	1.238
−12	.136	.000	−.136	1.569
−10	.214	.000	−.214	1.999
−8	.335	.001	−.334	2.565
−6	.524	.017	−.507	3.279
−4	.813	.107	−.706	4.036
−2	1.249	.383	−.866	4.667
0	1.894	.953	−.941	5.094
2	2.808	1.855	−.953	5.340
4	4.006	3.064	−.942	5.465
6	5.448	4.519	−.929	5.525
8	7.077	6.155	−.922	5.551
10	8.835	7.917	−.918	5.562
12	10.679	9.763	−.916	5.567
14	12.580	11.664	−.916	5.569
16	14.517	13.602	−.915	5.570
18	16.477	15.562	−.915	5.570
20	18.452	17.537	−.915	5.570
∞	∞	∞	−.915	5.570

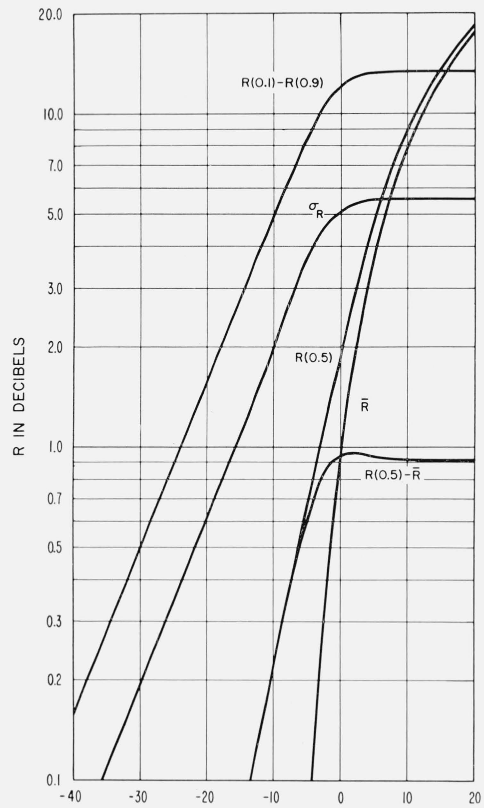

FIGURE 1. K *in decibels.*

and

$$\sigma_R \approx 4.3429\sqrt{1.6449-1/(2k^4)}. \tag{2.7}$$

$$\text{For } K<-20,\ \overline{R} \approx 4.3429k^2e^{-1/k^2} \tag{2.8}$$

$$\sigma_R \approx 6.1418k\sqrt{1+k^2/2}. \tag{2.9}$$

The magnitude of the error in these four expressions is less than $5(10)^{-4}$.

Values of the median, the mean, the difference between the mean and the median, and the standard deviation of R are given in table 1. These four quantities together with the interdecile range of R are shown as functions of K in figure 1. The median and interdecile range are taken from the paper by Norton et al. [1955].

3. Appendix 1. Derivation of the Formulas for $\overline{R}$ and σ_R

If we change to *natural* logarithms and make the substitution $r=ky$ in (2.1) and (2.2), we get

$$\overline{R}=2(20 \log_{10} e)e^{-1/k^2}[(\ln k)A(k)+B(k)], \tag{3.1}$$

and

$$\overline{R^2}=2(20 \log_{10} e)^2e^{-1/k^2}[(\ln k)^2A(k)+2(\ln k)B(k)+C(k)]. \tag{3.2}$$

Here

$$A(k)=\int_0^\infty ye^{-y^2}I_0(2y/k)dy=\left[\int_0^\infty y^{2a-1}e^{-y^2}I_0(2y/k)dy\right]_{a=1},$$

$$B(k)=\int_0^\infty (\ln y)ye^{-y^2}I_0(2y/k)dy=\left[\frac{\partial}{\partial a}\frac{1}{2}\int_0^\infty y^{2a-1}e^{-y^2}I_0(2y/k)dy\right]_{a=1},$$

and

$$C(k)=\int_0^\infty (\ln y)^2 y e^{-y^2} I_0(2y/k)dy=\left[\frac{\partial^2}{\partial a^2}\frac{1}{4}\int_0^\infty y^{2a-1}e^{-y^2}I_0(2y/k)dy\right]_{a=1}.$$

Using the Maclaurin series expansion for $I_0(2y/k)$ and integrating term by term, we find that

$$\int_0^\infty e^{-y^2}y^{2a-1}I_0(2y/k)dy=\frac{\Gamma(a)}{2}\,{}_1F_1(a;1;1/k^2),$$

where $\Gamma(x)$ is the gamma function and ${}_1F_1(a;c;x)$ is a confluent hypergeometric function [Rainville, 1960].

In order to complete the evaluation of $A(k)$, $B(k)$, and $C(k)$, we make use of the known series expansion for $-Ei(-x)$ and of the series transformation

$$e^{-x}\sum_{n=0}^\infty a_n\frac{x^n}{n!}=\sum_{n=0}^\infty b_n\frac{x^n}{n!},\text{ where } b_n=\Delta^n a_0 \tag{3.3}$$

is the nth forward difference of a_0; i.e., $\Delta^0 a_0=a_0$, $\Delta^1 a_0=a_1-a_0$, etc. Omitting any further details, we find that $\overline{R}$ is given by (2.3), and that

$$\overline{R^2}=(10\log_{10}e)^2\left\{4(\ln k)^2+4(\ln k)[\ln(1/k^2)-Ei(-1/k^2)]+\gamma^2\right.$$
$$\left.+\frac{\pi^2}{6}-2\gamma[\gamma+\ln(1/k^2)-Ei(-1/k^2)]+2\sum_{n=1}^\infty\frac{(-1)^{n+1}(1/k^2)^{n+1}}{(n+1)!(n+1)}\left(\sum_{j=1}^n\frac{1}{j}\right)\right\}. \tag{3.4}$$

Combining (2.3) and (3.4), we get (2.4) for σ_R.

In order to prove (2.5), we derive a different expression for the series in (3.4). Observe that

$$\sum_{n=1}^\infty\frac{(-1)^{n+1}(x)^{n+1}}{(n+1)!(n+1)}\left(\sum_{j=1}^n\frac{1}{j}\right)=\int_1^x\left[\sum_{n=1}^\infty\frac{(-1)^{n+1}t^n}{(n+1)!}\left(\sum_{j=1}^n\frac{1}{j}\right)\right]dt+C$$

where

$$C=\left[\sum_{n=1}^\infty\frac{(-1)^{n+1}x^{n+1}}{(n+1)!(n+1)}\left(\sum_{j=1}^n\frac{1}{j}\right)\right]_{x=1}=0.1827580.$$

Omitting any further details, except to note that use is made of the series expansions for $-Ei(-x)$ and $E^*(x)$, and that we again use the series transformation (3.3), we find that

$$\sum_{n=1}^\infty\frac{(-1)^{n+1}x^{n+1}}{(n+1)!(n+1)}\left(\sum_{j=1}^n\frac{1}{j}\right)=-Ei(-x)[E^*(x)-\gamma-\ln x]+Ei(-1)[E^*(1)-\gamma]$$
$$+\int_1^x\frac{e^t}{t}Ei(-t)dt-2\int_1^x\frac{Ei(-t)}{t}dt+\gamma\ln x+\frac{(\ln x)^2}{2}+C.$$

Using this expression together with (2.3) and (3.4), we arrive at (2.5).

4. Appendix 2. Evaluation of the Integrals $\int_1^{1/k^2}\frac{-Ei(-t)}{t}dt$ and $\int_1^{1/k^2}\frac{e^t}{t}[-Ei(-t)]dt$

The values of the integrals

$$g(1/k^2)=\int_1^{1/k^2}\frac{-Ei(-t)}{t}dt\text{ and }h(1/k^2)=\int_1^{1/k^2}\frac{e^t}{t}[-Ei(-t)]dt,$$

given in table 2 for various values of K, were obtained as follows. The ith entries of the table were computed in terms of the preceding entries by means of the formula

$$\int_1^{x_i}f(t)dt=\left(\int_1^{x_1}+\int_{x_1}^{x_2}+\ldots+\int_{x_{i-1}}^{x_i}\right)f(t)dt$$

TABLE 2

K	$1/k^2$	$g(1/k^2)$	$h(1/k^2)$
−0. 96910013	1. 2500000	0. 0404747	0. 123376
−2	1. 5848932	. 0679414	. 234919
−4	2. 5118864	. 0917103	. 402303
−6	3. 9810716	. 0971504	. 518481
−6. 9897000	5. 0000000	. 0976706	. 561387
−8	6. 3095733	. 0978116	. 597259
−9. 0308999	8. 0000000	. 0978393	. 626974
−10	10. 000000	. 0978428	. 649645
−10. 791812	12. 000000	. 0978432	. 665007
−12	15. 848932	. 0978432	. 683943
−14	25. 118864	. 0978432	. 706139
−16	39. 810716	. 0978432	. 720383
−18	63. 095734	. 0978432	. 729470
−20	100. 00000	. 0978432	. 735245
−25	316. 22776	. 0978432	. 742039
−30	1000. 0000	. 0978432	. 744196
−35	3162. 2776	. 0978432	. 744880
−40	10000. 000	. 0978432	. 745096
−∞	∞	. 0978432	. 745196

where each of the integrals on the right-hand side of the equation was evaluated numerically using a 16-point Gaussian quadrature formula. Eight significant digits were used in the calculations and the results were rounded to six significant digits.

A different method for calculating the values of the integrals corresponding to $K=-40$ is presented here to check the values given in table 2. Due to the technique used in preparing the table, this will also be a check on the accuracy of the remaining tabular entries. First observe that

$$\int_1^{10^4} \frac{e^t}{t}[-Ei(-t)]dt=\int_1^{\infty} \frac{e^t}{t}[-Ei(-t)]dt-0.0001+\epsilon$$

where $|\epsilon|<0.5(10)^{-8}$, and

$$\int_1^{10^4} \frac{-Ei(-t)}{t}\,dt=\int_1^{\infty} \frac{-Ei(-t)}{t}\,dt+\eta$$

where $|\eta|<10^{-10}$. If we use the Maclaurin series for e^t, we get

$$\int_1^{\infty} \frac{e^t}{t}[-Ei(-t)]dt=\int_1^{\infty} -\frac{Ei(-t)}{t}\,dt+\sum_{n=1}^{\infty}\frac{1}{n!}\int_1^{\infty} t^{n-1}[-Ei(-t)]dt.$$

From Erdélyi et al. [1953], we find that

$$\int_1^{\infty} t^{n-1}[-Ei(-t)]dt=\frac{\Gamma(n,1)+Ei(-1)}{n}=\frac{\Gamma(n)-\gamma(n,1)+Ei(-1)}{n}$$

where $\Gamma(n,x)$ and $\gamma(n,x)$ are the incomplete gamma functions, and from Le Caine [1948] we get

$$\int_1^{\infty} \frac{-Ei(-t)}{t}\,dt=\frac{\gamma^2+\frac{\pi^2}{6}}{2}-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)^3}.$$

Thus we find that

$$\int_1^{10^4} \frac{-Ei(-t)}{t}\,dt\approx 0.097843199$$

with an error less than $0.5(10)^{-9}$ in magnitude. In the same way we may also show that

$$\int_1^{12} \frac{-Ei(-t)}{t}\,dt\approx 0.0978432^{-}$$

with an error less than $0.5(10)^{-7}$ in magnitude. Omitting the details, we also get from the pre-

ceding equations that

$$\int_1^{10^4} \frac{e^t}{t}[-Ei(-t)]dt \approx 0.097843199 + \frac{\pi^2}{6} + Ei(-1)[E^*(1)-\gamma] - \sum_{n=1}^{\infty} \frac{\gamma(n, 1)}{n!n} - 0.0001 \approx 0.74509596$$

with an error less than $0.5(10)^{-8}$ in magnitude. It is seen that six significant digit accuracy is indicated for the values of the integrals given in table 2.

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